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A Two-Parameter Family of Elliptic Curves and Integer Solutions to the Quartic Diophantine Equation $A^4 + hB^4 = C^4 + hD^4$

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Abstract

We study the quartic Diophantine equation $A^4 + hB^4 = C^4 + hD^4$, where h is a fixed positive integer. Izadi and Baghalaghdam (2017) associated to each h a single elliptic curve $E(h)$ and showed that if $\mathrm{rank}E(h) > 0$, then the equation has infinitely many nontrivial primitive integer solutions. However, their method fails when $\mathrm{rank}E(h) = 0$ or when the generators of $E(h)(\mathbb{Q})$ have very large canonical height.

In this paper, we introduce a new rational parameter $k > 0$ and construct a two-parameter family of elliptic curves $E_{h,k}$. We prove that if $\mathrm{rank}E_{h,k} > 0$ for some k, then the equation has infinitely many nontrivial primitive integer solutions. This overcomes the limitations of the Izadi–Baghalaghdam method, as one can vary k to find curves with positive rank and manageable generator heights.

Assuming the Birch and Swinnerton-Dyer conjecture, we show that $\mathrm{rootno}(E_{h,k}) = -1$ implies $\mathrm{rank}E_{h,k} \geq 1$, providing a computable criterion for the existence of solutions. We present explicit computational examples, including $h = 20012, k = 2$ and $h = 20017, k = 1/4$, demonstrating the effectiveness of our method.

1 Introduction

The quartic Diophantine equation

$$A^4 + hB^4 = C^4 + hD^4, \tag{1}$$

where h is a fixed positive integer, has been studied by several authors. Izadi and Baghalaghdam [1] associated to each h a single elliptic curve $E(h)$ and proved that if $\mathrm{rank}E(h) > 0$, then equation (1) has infinitely many nontrivial primitive integer solutions (A, B, C, D).

However, their method has two serious limitations:

1. If $\mathrm{rank}E(h) = 0$, the method fails completely.
2. Even if $\mathrm{rank}E(h) > 0$, if the generators of $E(h)(\mathbb{Q})$ have very large canonical height, it is computationally infeasible to find explicit integer solutions.

In this paper, we overcome these limitations by introducing a new rational parameter $k > 0$. We construct a two-parameter family of elliptic curves $E_{h,k}$ and prove that if $\mathrm{rank}E_{h,k} > 0$ for some k, then equation (1) has infinitely many nontrivial primitive integer solutions. This allows us to vary k to find curves with positive rank and manageable generator heights.

Our main results are as follows:

- **Main Theorem:** If $\mathrm{rank} E_{h,k} > 0$, then equation (1) has infinitely many nontrivial primitive integer solutions.
- **Corollary (assuming BSD):** If $\mathrm{rootno}(E_{h,k}) = -1$, then $\mathrm{rank} E_{h,k} \geq 1$, and equation (1) has infinitely many nontrivial primitive integer solutions.
- **Explicit examples:** We present computational examples for $h = 20012, k = 2$ and $h = 20017, k = 1/4$, demonstrating the effectiveness of our method.

The rest of this paper is organized as follows. In Section 2, we review the Izadi–Baghalaghdam method and its limitations. In Section 3, we construct the two-parameter family $E_{h,k}$ and prove the main theorem. In Section 4, we assume the Birch and Swinnerton-Dyer conjecture and derive the corollary. In Section 5, we present explicit computational examples. In Section 6, we conclude with remarks and directions for future research.

2 Background: The Izadi–Baghalaghdam Method

2.1 The Izadi–Baghalaghdam Theorem

Izadi and Baghalaghdam [1] studied the quartic Diophantine equation

$$A^4 + hB^4 = C^4 + hD^4, \tag{2}$$

where h is a fixed rational number. They introduced the substitution

$$A = m - q, \quad B = m + p, \quad C = m + q, \quad D = m - p,$$

and derived the elliptic curve

$$E(h): \quad Y^2 = Z^3 - 3h^2 Z - (h^4 + h^2). \tag{3}$$

Their main theorem is as follows:

Theorem 2.1 (Izadi–Baghalaghdam [1])**.** *If* $\mathrm{rank} E(h) > 0$*, then equation* (2) *has infinitely many nontrivial primitive integer solutions* (A, B, C, D)*.*

2.2 Limitations of the Izadi–Baghalaghdam Method

The Izadi–Baghalaghdam method has two serious limitations:

1. **Rank zero problem:** The correspondence $h \mapsto E(h)$ is one-to-one. If $\mathrm{rank} E(h) = 0$, the method fails completely, as $E(h)(\mathbb{Q})$ consists only of torsion points.
2. **Large height problem:** Even if $\mathrm{rank} E(h) > 0$, if the generators of $E(h)(\mathbb{Q})$ have very large canonical height, it is computationally infeasible to find explicit rational points, and hence explicit integer solutions.

These limitations motivate the introduction of a new parameter k to construct a two-parameter family of elliptic curves.

3 Construction of the Two-Parameter Family $E_{h,k}$

3.1 Trivial and Nontrivial Solutions

We begin by clarifying the notion of trivial and nontrivial solutions to equation (1).

Definition 3.1 (Trivial and Nontrivial Solutions)**.** An integer solution (A, B, C, D) to the equation

$$A^4 + hB^4 = C^4 + hD^4$$

is called *trivial* if $A = \pm C$ and $B = \pm D$. Otherwise, it is called *nontrivial*.

Remark 3.2. Note that if (A, B, C, D) is a solution with $A = \pm C$, then the equation implies $hB^4 = hD^4$, and hence $B = \pm D$ (since $h > 0$). Thus, the condition "$A = \pm C$ and $B = \pm D$" is equivalent to "$A = \pm C$ or $B = \pm D$" for solutions to equation (1).

3.2 New Substitution

We introduce a new rational parameter $k > 0$ and the substitution

$$A = m - p, \quad B = km + q, \quad C = m + p, \quad D = km - q. \tag{4}$$

Substituting (4) into equation (1) and simplifying, we obtain

$$(hk^3p - q)m^2 = q^3 - hkp^3. \tag{5}$$

Without loss of generality, we may assume $hk^3p - q = 1$, so that $q = hk^3p - 1$. Substituting this into (5), we get

$$m^2 = (hk^3p - 1)^3 - hkp^3. \tag{6}$$

Expanding the right-hand side of (6), we obtain

$$m^2 = (h^3k^9 - hk)p^3 - 3h^2k^6p^2 + 3hk^3p - 1. \tag{7}$$

3.3 Elliptic Curve $E_{h,k}$

Multiplying both sides of (7) by $(h^3k^9 - hk)^2$ and setting

$$X = (h^3k^9 - hk)p, \quad Y = (h^3k^9 - hk)m,$$

we obtain the elliptic curve

$$E_{h,k}: \quad Y^2 = X^3 - 3h^2k^6X^2 + 3hk^3(h^3k^9 - hk)X - (h^3k^9 - hk)^2. \tag{8}$$

Remark 3.3. When $k = 1$, the elliptic curve $E_{h,k}$ reduces to

$$E_{h,1}: \quad Y^2 = X^3 - 3h^2X^2 + 3h(h^3 - h)X - (h^3 - h)^2.$$

After the change of variables $X = Z + h^2$, this becomes

$$Y^2 = Z^3 - 3h^2Z - (h^4 + h^2),$$

which is precisely the Izadi–Baghalaghdam curve $E(h)$ given by (3). Thus, our construction can be viewed as a natural generalization of the Izadi–Baghalaghdam method, where the parameter k provides additional flexibility in finding curves with positive rank.

Theorem 3.4. *The elliptic curve $E_{h,k}$ is singular if and only if $hk = 0$ or $hk^4 = \pm 1$.*

Proof. The discriminant of $E_{h,k}$ is given by

$$\mathrm{Disc}(E_{h,k}) = -432h^4k^4(hk^4 - 1)^2(hk^4 + 1)^2. \tag{9}$$

An elliptic curve is singular if and only if its discriminant is zero. From (9), we have $\mathrm{Disc}(E_{h,k}) = 0$ if and only if at least one of the following holds:

1. $h = 0$,
2. $k = 0$,
3. $hk^4 - 1 = 0$, i.e., $hk^4 = 1$,
4. $hk^4 + 1 = 0$, i.e., $hk^4 = -1$.

Thus, $E_{h,k}$ is singular if and only if $hk = 0$ or $hk^4 = \pm 1$. □

Corollary 3.5. *Let $h, k \in \mathbb{Q}$ with $h, k > 0$. If $hk^4 - 1 \neq 0$, then the elliptic curve $E_{h,k}$ is non-singular.*

Proof. This follows immediately from Theorem 3.4. Since $h, k > 0$, we have $hk \neq 0$. Furthermore, $hk^4 - 1 \neq 0$ implies $hk^4 \neq 1$. Finally, since $h, k > 0$ and $k^4 > 0$, we have $hk^4 > 0$, so $hk^4 \neq -1$.

Thus, none of the singularity conditions in Theorem 3.4 hold, and $E_{h,k}$ is non-singular. □

Remark 3.6. In our main theorem, we assume h is a positive integer and $k \in \mathbb{Q}_{>0}$, so $hk > 0$ is automatically satisfied. Thus, we only need to ensure $hk^4 \neq 1$ to guarantee that $E_{h,k}$ is a non-singular elliptic curve.

3.4 Nontriviality of Solutions

We now show that our construction always yields nontrivial solutions from non-torsion points.

Theorem 3.7. *Let $h, k \in \mathbb{Q}$ with $h, k > 0$. If $(X, Y) \in E_{h,k}(\mathbb{Q})$ is a non-torsion point, then the integer solution (A, B, C, D) obtained from (X, Y) via the inverse transformation is nontrivial.*

Proof. We prove the contrapositive: if (A, B, C, D) is a trivial solution, then (X, Y) corresponds to a torsion point (or is undefined).

Assume (A, B, C, D) is a trivial solution. By Definition 3.1, we have $A = \pm C$ and $B = \pm D$.

From the substitution (4), we compute:

$$C^2 - A^2 = (m + q)^2 - (m - q)^2 = 4mq,$$

$$B^2 - D^2 = (km + p)^2 - (km - p)^2 = 4kmp.$$

If $A = \pm C$ and $B = \pm D$, then we must have $mq = 0$ and $kmp = 0$. Since $k > 0$, we obtain $m = 0$ or $p = q = 0$.

Case $m = 0$: The associated rational point (X,Y) is a torsion point of $E_{h,k}$ with $Y = 0$.

Case $p = q = 0$: From our construction, this contradicts the relation

$$hk^3p - q = 1. \tag{10}$$

Hence, by contraposition, if (X, Y) is a non-torsion point, then (A, B, C, D) is a nontrivial solution. □

Remark 3.8. This theorem ensures that our construction always yields nontrivial solutions from non-torsion points on $E_{h,k}$.

3.5 Main Theorem

We now state and prove our main theorem.

Theorem 3.9 (Main Theorem)**.** *Let h be a positive integer and $k \in \mathbb{Q}_{>0}$ with $hk^4 \neq 1$. If* $\mathrm{rank} E_{h,k} > 0$*, then the Diophantine equation*

$$A^4 + hB^4 = C^4 + hD^4$$

has infinitely many nontrivial primitive integer solutions (A, B, C, D).

Proof. Assume $\mathrm{rank} E_{h,k} > 0$. By Corollary 3.5, $E_{h,k}$ is a non-singular elliptic curve. By the Mordell–Weil theorem, $E_{h,k}(\mathbb{Q})$ is a finitely generated abelian group:

$$E_{h,k}(\mathbb{Q}) \cong \mathbb{Z}^r \oplus T,$$

where $r = \mathrm{rank} E_{h,k} > 0$ and T is the torsion subgroup.

Since $r > 0$, there exist infinitely many non-torsion rational points $(X, Y) \in E_{h,k}(\mathbb{Q})$. For each such point, we can recover p, q, m via the inverse transformation:

$$p = \frac{X}{h^3k^9 - hk}, \tag{11}$$

$$m = \frac{Y}{h^3k^9 - hk}, \tag{12}$$

$$q = hk^3p - 1. \tag{13}$$

Substituting these into (4), we obtain rational solutions $(A, B, C, D) \in \mathbb{Q}^4$. By Theorem 3.7, these solutions are nontrivial.

Multiplying by an appropriate rational number $t \in \mathbb{Q}^\times$, we can clear denominators and obtain integer solutions $(tA, tB, tC, tD) \in \mathbb{Z}^4$. Finally, we can normalize so that $0 < A < C$, $0 < D < B$, and $\gcd(A, B, C, D) = 1$.

Since there are infinitely many rational points (X, Y), there are infinitely many nontrivial primitive integer solutions (A, B, C, D). □

Remark 3.10. The key advantage of our method over the Izadi–Baghalaghdam method is that we have a two-parameter family $(h, k) \mapsto E_{h,k}$, rather than a one-to-one correspondence $h \mapsto E(h)$. This allows us to vary k to find curves with positive rank and manageable generator heights.

4 Corollary Assuming the Birch and Swinnerton-Dyer Conjecture

4.1 Root Number and Rank

Let $E/\mathbb{Q}$ be an elliptic curve and $L(E, s)$ its Hasse–Weil L-function. The Birch and Swinnerton-Dyer conjecture states that

$$\mathrm{rank} E(\mathbb{Q}) = \mathrm{ord}_{s=1} L(E, s).$$

The L-function satisfies the functional equation

$$\Lambda(E, s) = \mathrm{rootno}(E) \cdot \Lambda(E, 2 - s),$$

where $\mathrm{rootno}(E) \in \{+1, -1\}$ is the root number (sign) of E.

If $\mathrm{rootno}(E) = -1$, then the functional equation implies that $L(E, s)$ has a zero at $s = 1$ of odd order. Assuming the Birch and Swinnerton-Dyer conjecture, we have

$$\mathrm{rootno}(E) = -1 \quad \Longrightarrow \quad \mathrm{rank} E(\mathbb{Q}) \geq 1.$$

4.2 Corollary

Applying this to our two-parameter family $E_{h,k}$, we obtain the following corollary.

Corollary 4.1 (Assuming BSD)**.** *Assume the Birch and Swinnerton-Dyer conjecture. If* $\mathrm{rootno}(E_{h,k}) = -1$*, then* $\mathrm{rank}E_{h,k} \geq 1$*, and the Diophantine equation*

$$A^4 + hB^4 = C^4 + hD^4$$

has infinitely many nontrivial primitive integer solutions (A, B, C, D)*.*

Proof. If $\mathrm{rootno}(E_{h,k}) = -1$, then the functional equation implies $\mathrm{ord}_{s=1}L(E_{h,k}, s) \geq 1$. Assuming the Birch and Swinnerton-Dyer conjecture, we have $\mathrm{rank}E_{h,k} = \mathrm{ord}_{s=1}L(E_{h,k}, s) \geq 1$. By Theorem 3.9, the equation has infinitely many nontrivial primitive integer solutions. □

Remark 4.2. The root number $\mathrm{rootno}(E_{h,k})$ is computable from the local data of $E_{h,k}$ (conductor and local factors). Thus, Corollary 4.1 provides a computable criterion for the existence of solutions.

5 Explicit Computational Examples

In this section, we present explicit computational examples demonstrating the effectiveness of our method. All computations were performed using the Magma computer algebra system [2].

5.1 Example 1: $h = 20012, k = 2$

For $h = 20012$ and $k = 2$, the elliptic curve $E_{20012,2}$ is given by

$$E_{20012,2}: \quad Y^2 = X^3 - 76892187648X^2 + 19708028404125248770 56X - 16837704646748128716194298682944.$$

The minimal standard model is

$$E^{\mathrm{mstd}}_{20012,2}: \quad Y^2 = X^3 - 1201440432X - 2566149531837201785.$$

The root number is $\mathrm{rootno}(E^{\mathrm{mstd}}_{20012,2}) = -1$. Assuming BSD, we have $\mathrm{rank}E_{20012,2} \geq 1$.

Using Magma's 4-descent, we find a generator of $E^{\mathrm{mstd}}_{20012,2}(\mathbb{Q})$:

$$P_1 = \left(\frac{69990011273486750657}{658168748176}, \frac{5855360096630063991688276304 97}{533956509345232576} \right),$$

with canonical height $\mathrm{height}(P_1) \approx 38.81$.

Transforming back to $E_{20012,2}(\mathbb{Q})$, we obtain

$$Q_1 = \left(\frac{4287326252006674228161}{164542187044}, \frac{5855360096630063991688276304 97}{66744563668154072} \right),$$

with $\mathrm{height}(Q_1) \approx 38.81$.

From $Q_1, [2]Q_1, [3]Q_1$, we obtain the following nontrivial primitive integer solutions to $A^4 + 20012B^4 = C^4 + 20012D^4$:

$$1224358877541330213614366 3^4 + 20012 \cdot 3626402239183894949300168^4 = 1586461360566234529306772 9^4 + 20012 \cdot 3615647421314191364547964^4,$$

$$869479390900569207583808478574665811522008883566216732063691452483876275444611811360760888231975936189^4$$
$$+ 20012 \cdot 260420093920740443113185151067694545557713490001825666040158525796184319722116456703148779799870281258^4$$
$$= 212687879746867515920361007570250309648899031763036489820884984489320772380761655991861295151729113779874$$
$$+ 20012 \cdot 254378719215471460126418043187980024376249378126470666250158259022478577003893040412555346882000602338^4,$$

and a third solution (omitted for brevity).

5.2 Example 2: $h = 20017, k = 1/4$

For $h = 20017$ and $k = 1/4$, the elliptic curve $E_{20017,1/4}$ is given by

$$E_{20017,1/4}: \quad Y^2 = X^3 - \frac{799908723}{4096}X^2 + \frac{213232232225759715}{16777216}X - \frac{1894715549104816212938302 5}{68719476736}.$$

The minimal standard model is

$$E^{\mathrm{mstd}}_{20017,1/4}: \quad Y^2 = X^3 - 307722461952X - 2569135247624694800.$$

The root number is $\mathrm{rootno}(E^{\mathrm{mstd}}_{20017,1/4}) = +1$. Using Magma's 4-descent, we find two generators of $E^{\mathrm{mstd}}_{20017,1/4}(\mathbb{Q})$:

$$P_1 = \left(\frac{82650193}{36}, \frac{641685909799}{216}\right), \quad \mathrm{height}(P_1) \approx 5.88,$$
$$P_2 = \left(\frac{458484600937}{16641}, \frac{310364754847345747}{2146689}\right), \quad \mathrm{height}(P_2) \approx 20.28.$$

Thus, $\mathrm{rank} E_{20017,1/4} \geq 2$.

Transforming back to $E_{20017,1/4}(\mathbb{Q})$, we obtain

$$Q_1 = \left(\frac{3936723373}{36864}, \frac{641685909799}{884736}\right), \quad \mathrm{height}(Q_1) \approx 5.88,$$
$$Q_2 = \left(\frac{14003474304241}{68161536}, \frac{310364754847345747}{8792838144}\right), \quad \mathrm{height}(Q_2) \approx 20.28.$$

From $[i]Q_1 + [j]Q_2$ with $\max\{|i|, |j|\} \leq 2$, we obtain the following nontrivial primitive integer solutions to $A^4 + 20017B^4 = C^4 + 20017D^4$:

$$1874^4 + 20017 \cdot 259^4 = 3178^4 + 20017 \cdot 67^4,$$
$$66290484208883^4 + 20017 \cdot 6364824463927^4 = 84914548787659^4 + 20017 \cdot 1708808319233^4,$$
$$2866927411747^4 + 20017 \cdot 15866041244359^4 = 121173540129781^4 + 20017 \cdot 15144075641023^4,$$

and eight additional solutions (omitted for brevity).

Remark 5.1. Note that the first solution $1874^4 + 20017 \cdot 259^4 = 3178^4 + 20017 \cdot 67^4$ is very small, demonstrating that our method can find solutions with manageable sizes.

6 Conclusion and Future Directions

In this paper, we introduced a two-parameter family of elliptic curves $E_{h,k}$ and proved that if $\mathrm{rank}E_{h,k} > 0$, then the quartic Diophantine equation $A^4 + hB^4 = C^4 + hD^4$ has infinitely many nontrivial primitive integer solutions. This overcomes the limitations of the Izadi–Baghalaghdam method, as one can vary k to find curves with positive rank and manageable generator heights.

Assuming the Birch and Swinnerton-Dyer conjecture, we showed that $\mathrm{rootno}(E_{h,k}) = -1$ implies $\mathrm{rank}E_{h,k} \geq 1$, providing a computable criterion for the existence of solutions. We presented explicit computational examples, including $h = 20012, k = 2$ and $h = 20017, k = 1/4$, demonstrating the effectiveness of our method.

There are several directions for future research:

1. **Explicit formula for the root number:** Derive an explicit formula for $\mathrm{rootno}(E_{h,k})$ in terms of h and k. This would allow us to predict which (h, k) give root number -1.
2. **Existence of rank-positive k:** Prove that for every positive integer h, there exists $k \in \mathbb{Q}_{>0}$ such that $\mathrm{rank}E_{h,k} > 0$.
3. **Statistical analysis of heights:** Analyze the distribution of canonical heights of generators as k varies, and develop heuristics for finding k with small generator heights.
4. **Generalization to other quartic equations:** Apply our method (introducing an additional parameter k) to other quartic Diophantine equations.

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