Constructing Infinite Families of Solutions to $x^4 + y^4 + z^4 = 2n^2w^4$ with $z \neq x + y$ via 4-descent on Elliptic Curves

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Abstract

This paper presents a method to construct infinite families of primitive integer solutions ($\gcd(x, y, z, w) = 1$ and $z \neq x + y$) to the quartic Diophantine equation

$$x^4 + y^4 + z^4 = 2n^2w^4$$

for square-free positive integers n satisfying $\gcd(n, 290) = 1$, using rational points on elliptic curves.

We reduce the problem to common rational points on parameter-dependent quadratic curves (4), (5), and further to a quartic elliptic curve (7) via the chord-and-tangent method. MAGMA's 4-descent is employed to compute the Mordell-Weil group, generating infinite families of solutions. Local congruence conditions are shown to be necessary and sufficient for solution existence, with singular cases handled elementarily.[4, 5]

Specifically, we provide non-trivial solutions for $n = 101, 1007$ where the canonical heights of the corresponding rational points exceed 210, demonstrating the effectiveness of the 4-descent approach.

1 Introduction

The study of quartic Diophantine equations of the form $x^4 + y^4 + z^4 = Dw^4$ has been central in number theory since Noam Elkies' landmark discovery of counterexamples to Euler's conjecture [1]. While Elkies found solutions to $x^4 + y^4 + z^4 = w^4$, the case with coefficient $2n^2$ on the right remains largely unexplored except for computational searches [8, 9, 10].

This paper provides the first parametrization of primitive solutions ($\gcd(x, y, z, w) = 1$ and $z \neq x + y$) for several square-free n satisfying specific local conditions, via a novel reduction to elliptic curve rational points.

Using the above parameterization method, we have found infinite families of integer solutions to the Diophantine equation $x^4 + y^4 + z^4 = 2n^2w^4$ with $z \neq x + y$ for each $n = 33$, 41, 47, 51, 59, 69, 77, 83, 89, 101, 119, 123, 137, 141, 149, 159, 161, 173, 179, 187, 197, 209, 213, 227, 233, 383, 389, 393, 1007 and 1013, for which no previously known solutions exist, in the range $n \leq 233$.

In this study, we verify the validity of the Hasse's principle for specific n, and by using the 4-descent of MAGMA, we prove that the elements of the Tate-Shafarevich group actually correspond to rational points in concrete numbers.

2 Problem Statement and Conditions

Consider the equation

$$x^4 + y^4 + z^4 = 2n^2w^4 \tag{1}$$

where n is square-free and satisfies:

$$\begin{aligned} 2 &\nmid n, \\ 5 &\nmid n, \\ 29 &\nmid n. \end{aligned} \tag{2}$$

We can easily confirm that if n is square-free and $\gcd(n, 290) > 1$, then the equation (1) have no solutions with $\gcd(x, y, z, w) = 1$ by checking $\pmod{16}$, $\pmod{5}$, and $\pmod{29}$.

Assuming $z \neq 0$, normalize by dividing by z^4, and replace $\frac{x}{z}, \frac{y}{z}, \frac{w}{z}$ to x, y, t:

$$x^4 + y^4 + 1 = 2n^2t^4. \tag{3}$$

3 Quadratic Curves and Common Points

Introduce rational parameter u and two quadratic curves:

$$(u^2 - 2)y^2 = (-u^2 + 4u - 2)x^2 - 2(u^2 - 2u + 2)x + (-u^2 + 4u - 2) \tag{4}$$

$$\pm n(u^2 - 2)t^2 = (u^2 - 2u + 2)x^2 + (-u^2 + 4u - 2)x + (u^2 - 2u + 2). \tag{5}$$

Theorem 1. Common rational solutions (x, y, t) to (4),(5) yield solutions to (3).[4]

4 The Involution Map

For rational parameter u and variables x, y, t, we deine the following involution map τ:

$$\tau(u, x, y, t) = \begin{cases} (\frac{u-1}{u-2}, -x, y, t) & \text{if } u \neq 2, \\ (2, -x, y, t) & \text{if } u = 2 \text{ .} \end{cases} \tag{6}$$

The involution τ preserves the quadratic curves (4) and (5), and therefore all their common rational points (x, y, t) with except for the sign of x.

Therefore, the integer solutions of (1) are identical for parameter u and parameter $\tau(u) = \frac{u-1}{u-2}$. In other words, it is sufficient to find the common rational point (x, y, t) for either parameter u or parameter $\tau(u)$.

5 Reduction to Quartic Elliptic Curve

Given a constant H (for example, 200), using Helmut Hasse's local-global principle, we can select a rational number u such that the height $h(u) \leq H$, so that the two quadratic curves (4) and (5) each have a rational point. Then, we can filter the rational parameter u effectively. This contributes to the efficiency of rational point search.

For each rational parameter u, fix initial rational point (x_0, y_0) on (4); intersect with line

$$y = k(x - x_0) + y_0$$

to get second point $(x(k), y(k))$ as rational functions of k.

Substitute $x(k)$ into (5) to obtain the assoicated elliptic curve:

$$E_{n,u} : \pm Y^2 = aX^4 + bX^3 + cX^2 + dX + e, \tag{7}$$

where $Y = t(pk^2 + qk + r)^2$, $X = k$, and $p, q, r, a, b, c, d, e \in \mathbb{Z}$.

The right-hand quadratic in (5) has no real roots, so sign is unique: $+Y^2$ if $a > 0$, $-Y^2$ if $a < 0$.[5]

Because $a, b, c, d, e \in \mathbb{Z}$, a rational point (X, Y) on (7) with $X \notin \mathbb{Z}$ should be a non-torsion point .

6 Singular Cases

The first quadratic curve (4) is singular iff $u = 0, 1, 2$; then reduces to

$$x^2 + x + 1 = nt^2,$$

solvable only for $n = 1$. The second quadratic curve (5) is always nonsingular.[4]

Seiji Tomita and Oliver Couto [2] (Th.4.2) give a parameter solution to (1) using polynomials in p and q if there exist integers p and q such that $n = 3p^2 + q^2$. However, our parameter solution via the elliptic curves is different from this polynomial solution and we could apply even when $n \neq 3p^2 + q^2$.

7 Computing Rational Points with MAGMA

Transform (7) to Weierstrass form; apply MAGMA's `FourDescent` to compute rank, generators, and torsion.

We choose a rational parameter u and the associated elliptic curve (7) with positive rank . For non-torsion point (X, Y) on the curve (7) , generate m-multiples; set $k = X$ and back-substitute to (x, y, t), then clear denominators for primitive (x, y, z, w).[5]

8 Necessity and Sufficiency of Local Conditions

Theorem 2. For arbitary square-free integer n, infinite primitive solutions of (1) exist iff $\gcd(n, 290) = 1$ and there exist a rational number u and the associated elliptic curve $E_{n,u}$ with positive rank .

Proof: Sufficiency via $rank E_{n,u} \geq 1$; necessity via p-adic obstructions for p=2,5,29.[3, 4]

Conjecture 1. If n is a square-free integer and $\gcd(n, 290) = 1$, then $rank E_{n,u} > 0$ for some rational number u.

Remark 1. Conjecture 1 is valid for $n \leq 247$.

9 Numerical Examples

For example $n = 41$, we can filter rational number u with $height(u) \leq 200$ so that both quadrtic curve (4) and (5) has a rational point. Then we can examine the following 144 rational numbers u:

$$
\begin{aligned}
u = &-1, \frac{1}{5}, \frac{1}{17}, \frac{-1}{49}, \frac{1}{53}, \frac{3}{2}, \frac{4}{17}, \frac{-4}{45}, \frac{-4}{49}, \frac{4}{137}, \frac{7}{9}, \frac{8}{9}, \frac{8}{25}, \frac{-8}{45}, \frac{8}{81}, \\
&\frac{-8}{101}, \frac{9}{4}, 10, \frac{11}{2}, \frac{-11}{149}, \frac{12}{109}, \frac{-12}{185}, \frac{13}{53}, \frac{15}{17}, \frac{15}{113}, \frac{-16}{97}, \frac{17}{81}, \frac{19}{2}, \frac{19}{149}, \frac{-19}{193}, \\
&\frac{-20}{81}, \frac{20}{109}, \frac{21}{37}, \frac{21}{61}, \frac{-28}{153}, \frac{29}{49}, \frac{30}{13}, \frac{-32}{53}, \frac{-32}{117}, \frac{3}{16}, \frac{-37}{181}, \frac{40}{113}, \frac{42}{17}, \frac{-43}{153}, \frac{44}{169}, \\
&\frac{48}{193}, \frac{-51}{157}, \frac{-52}{73}, \frac{52}{137}, \frac{3}{16}, \frac{-55}{49}, \frac{-56}{89}, \frac{56}{145}, \frac{57}{109}, \frac{61}{113}, \frac{-61}{117}, \frac{-64}{49}, \frac{64}{153}, \frac{-64}{157}, \frac{-64}{169}, \\
&\frac{-67}{61}, \frac{-67}{81}, \frac{67}{101}, \frac{69}{20}, \frac{76}{113}, \frac{-76}{153}, \frac{-76}{193}, \frac{76}{197}, \frac{77}{109}, \frac{-79}{153}, \frac{-80}{149}, \frac{-84}{65}, \frac{-84}{101}, \frac{88}{117}, \frac{-88}{145}, \\
&\frac{89}{121}, \frac{91}{149}, \frac{-91}{181}, \frac{92}{157}, \frac{93}{40}, \frac{94}{49}, \frac{-95}{149}, \frac{98}{53}, \frac{99}{50}, \frac{100}{153}, \frac{101}{40}, \frac{-101}{117}, \frac{101}{121}, \frac{102}{53}, \frac{-104}{125}, \\
&\frac{105}{52}, \frac{-112}{117}, \frac{116}{121}, \frac{-116}{149}, \frac{-116}{173}, \frac{-119}{89}, \frac{119}{145}, \frac{-120}{101}, \frac{120}{169}, \frac{126}{5}, \frac{127}{153}, \frac{128}{145}, \frac{-128}{153}, \frac{-132}{97}, \frac{-132}{181}, \\
&\frac{135}{34}, \frac{138}{85}, \frac{141}{20}, \frac{141}{32}, \frac{145}{64}, \frac{146}{29}, \frac{147}{157}, \frac{150}{37}, \frac{-152}{197}, \frac{153}{32}, \frac{153}{104}, \frac{154}{73}, \frac{161}{52}, \frac{161}{169}, \frac{162}{17}, \\
&\frac{162}{113}. \frac{165}{52}, \frac{167}{10}, \frac{168}{173}, \frac{171}{26}, \frac{173}{181}, \frac{-176}{197}, \frac{177}{8}, \frac{-177}{193}, \frac{177}{197}, \frac{178}{5}, \frac{179}{26}, \frac{182}{101}, \frac{186}{73}, \frac{187}{197}, \\
&\frac{-188}{137}, \frac{-188}{185}, \frac{189}{8}, \frac{189}{128}, \frac{-196}{153}, \frac{-196}{181}, \frac{198}{89}, \frac{198}{125}, \frac{-199}{193}.
\end{aligned}
$$

For $u = \frac{198}{125}$, we easily get a inital rational point $R(\frac{-15559}{420776}, \frac{775755}{420776})$ of the quadratic curve (4).

A line with a slope of k that passes through R intersects the quadratic curve (4) with another point $(x(k).y(k))$.

Then, we have following:

$$
\begin{aligned}
y =& k\left(x+\frac{15559}{420776}\right)+\frac{775755}{420776}, && (8)\\
x(k) =& \frac{61878143k^2+6170355270k-8594866697}{-1673426152k^2+6005735848}, && (9)\\
y(k) =& \frac{3085177635k^2-8372793090k+11072351115}{-1673426152k^2+6005735848}. && (10)
\end{aligned}
$$

From (9) and (5), we have followings:

$$
\begin{aligned}
\left\{17251816(3977k^2-14273)^2t\right\}^2 =& 287644205070868215349k^4-1436879617899897971340k^3\\
&+3997420003135110578878k^2-6003499258764516668940k+4279348392455021754229.
\end{aligned} \tag{11}
$$

We set

$$
\begin{aligned}
Y =& 17251816(3977k^2-14273)^2t, && (12)\\
X =& k, && (13)
\end{aligned}
$$

then we have following the associated elliptic curve

$$
\begin{aligned}
E_{41,u} : Y^2 =& 287644205070868215349X^4-1436879617899897971340X^3+3997420003135110578878X^2\\
&-6003499258764516668940X+4279348392455021754229
\end{aligned} \tag{14}
$$

We confirm the syzygy maps an elliptic curve

$$
\begin{aligned}
E^0_{41,u} : y^2 =& x^3-131533381367244164215248081536630341171740672x\\
&-496946952622325121564854492450656958760656582555086171408919691264
\end{aligned} \tag{15}
$$

to the elliptic curve $E_{41,u}$.
The minimal standard model of $E^0_{41,u}$ is

$$
E^2_{41,u} : y^2 = x^3+x^2-3237629437155959990x-1919096286007227296033058600. \tag{16}
$$

Considering $E_{41,u}$ to 2-descent of $E^2_{41,u}$, we execute MAGMA's 4-descent, and get two independent rational points of $E^2_{41,u}$:

$$
\begin{aligned}
&P_1\left(\frac{-174942651055478264675987}{244780606357156}, \frac{20891124644686233492499046337781371}{3829706654220268054696}\right),\\
&P_2\left(\frac{-14294068250419325295136330646500671045398507}{10710488752832501604605089844433796},\right.\\
&\left.\frac{5513451734757779144704564173941245101175055697275239743153358521}{1108444452616653816973134639524599540568006219348344}\right).
\end{aligned}
$$

By rational transformation $[2524656, 2124629306112, 0, 0]^{-1}$. we get two independent rational points of $E^0_{41,u}$

$$
\begin{aligned}
&Q_1\left(\frac{-278766212361028884962560363157615040}{61195151589289},\right.\\
&\left.\frac{42022169170381781318268794555009598687378195785697792}{478713331777533506837}\right),\\
&Q_2\left(\frac{-22777197225615531144367809302749081158104851178197540800}{2677622188208125401151272461108449},\right.\\
&\left.\frac{11090221586976991451411531078086682941672735310414994171896001971993851427077934592}{138555556577081727121641829940574942571000777418543}\right).
\end{aligned}
\tag{17}
$$

By syzygy map, We have some rational points (X, Y) of $E_{41,u}$ from rational points of $E^0_{41,u}$, and some $k = X$:

$$
\begin{aligned}
k =&\frac{30407075}{26279287}, \frac{945967865}{1248057599}, \frac{1588437129}{13721456335}, \frac{-1219384020107}{2053661177915},\\
&\frac{7238621986098507}{15401438000287805}, \frac{7238621986098507}{15401438000287805}, \frac{2523484525026218281}{616919842550050055}, \frac{2523484525026218281}{616919842550050055},\\
&\frac{7918680427205562475}{1555531906528806611}, \frac{7918680427205562475}{1555531906528806611}, \frac{206736470097243783595}{455301048052810772797}, \frac{206736470097243783595}{455301048052810772797},\\
&\cdots .
\end{aligned}
$$

From (9),(10),(14),(12).(13), we have a rational point (x, y, t) which satifies (3). We have an integer solution (x, y, z, w) with $\gcd(x, y, z, w) = 1$ which satisfies (1). Changing sign and replacing x, y, z so that $0 < x \le y \le z$, we have following equations:

$$
\begin{aligned}
&35401855^4 + 40865628^4 + 53562031^4 = 3362 \cdot 7822733^4 \\
&762983397233069 40^4 + 144376098024837517^4 + 392097054273222611^4 = 3362 \cdot 51745745604910607^4 \\
&22630934278564908444565^4 + 21842600916841019342451 6^4 + 38343047000803923412388 3^4 \\
&= 3362 \cdot 51630869931062938919159^4 \\
&66362246478987836342552992091201664685210465639474020^4 \\
&+ 96903091845971137820040019904874448516482466179139001^4 \\
&+ 124994312898188111523490811243725150204657332635449303^4 \\
&= 3362 \cdot 179838831232746850498471369045331511826771885565115 41^4
\end{aligned}
$$

For $u = \frac{-52}{137}$, similarly, we have the associated elliptic curve

$$
\begin{aligned}
E_{41,u} : Y^2 =& 146130108644622127X^4 - 345859546045705 60X^3 + 75769621840114278X^2 \\
&- 370651588684471816X + 306037888200539443
\end{aligned} \tag{18}
$$

and rational numbers $k = X$:

$$
\begin{aligned}
k =& \frac{16427}{15371}, \frac{-74303}{29149}, \frac{1042559}{1271441}, \frac{-2229773}{1793951}, \\
& \frac{934714950429932604431377977}{651572367992463462669439919}, \frac{-618926210953415209621494069 17}{557172195793843534777359955 9}, \\
& \frac{1069338108481668757405299870337}{1855015185625408790131546702081}, \frac{1277338685423332040496182410519}{2234155375514514753340048721189}, \\
& \cdots .
\end{aligned}
$$

and integer solutions of (1)

$$
\begin{aligned}
&92988^4 + 185585^4 + 200711^4 = 3362 \cdot 30433^4 \\
&31462395618155335427324997209975564596059352209401 3^4 \\
&+ 33830232913875010523309417936289725557988600411726621^4 \\
&+ 58958393745878311529280931462685062088878588503601020^4 \\
&= 3362 \cdot 79445726929376059312175416967121359045209773084337 27^4 \\
&\cdots .
\end{aligned}
$$

For $u = \frac{44}{169}$, similarly, we have the associated elliptic curve

$$
\begin{aligned}
E_{41,u}: Y^2 =& 417877855555951071X^4 + 1299785550450804690X^3 + 1519574898215554612X^2 \\
&+ 791044232470008790X + 154651753418222341
\end{aligned} \tag{19}
$$

and rational numbers $k = X$:

$$
\begin{aligned}
k =& \frac{-36225655093}{42531504399}, \frac{-737917016899}{985152753743}, \frac{-3284931787843}{315094317701}, \frac{-14199170625974}{19434669271167}, \\
&\cdots.
\end{aligned}
$$

and integer solutions of (1)

$$
\begin{aligned}
&544138968637796619 7^4 + 5952881649156946338 1^4 + 7370607112236248198 0^4 \\
&\quad = 3362 \cdot 10576637809123172967^4 \\
&\cdots.
\end{aligned}
$$

Similarly, for several n we can find a rational number u, a initial rational point (x_0, y_0) of (4), a rational number k , and an integer solution (x, y, z, w) with $0 < x \leq y \leq z$, the result is as follows:

n	u	(x_0, y_0)	k	Solution (x, y, z, w) $(0 < x \leq y \leq z)$	height(k)
1	$\frac{938}{241}$	$\left(\frac{1799}{1172}, \frac{1565}{1172}\right)$	$\frac{30407075}{26279287}$	(11270111669357, 22338600682595, 80267274165144, 67603989724187)	67.012
3	$\frac{-504}{4817}$	$\left(\frac{-2566}{1609}, \frac{4519}{1609}\right)$	$\frac{-4519}{957}$	(1609, 2566, 4519, 2257)	21.068
33	$\frac{24}{53}$	$\left(\frac{-309}{763}, \frac{100}{109}\right)$	$\frac{14467817}{5083051}$	(528988010581, 673826751736, 822834434251, 135897934731)	56.375
41	$\frac{198}{125}$	$\left(\frac{-15559}{420776}, \frac{775755}{420776}\right)$	$\frac{2558149}{3980783}$	(35401855, 40865628, 53562031, 7822733)	36.996
47	$\frac{32}{85}$	$\left(\frac{-165}{4573}, \frac{-20158}{32011}\right)$	$\frac{1262}{987}$	(9051, 142546, 264089, 33059)	24.724
51	$\frac{-73}{121}$	$\left(\frac{835}{1143}, \frac{-1298}{1143}\right)$	$\frac{-553}{1283}$	(129205, 145309, 168674, 23303)	22.230
59	$\frac{53}{345}$	$\left(\frac{6408}{9377}, \frac{3521}{9377}\right)$	$\frac{6419}{7003}$	(228599, 398665, 545334, 63949)	27.057
69	$\frac{5}{169}$	$\left(\frac{182033}{581575}, \frac{379548}{581575}\right)$	$\frac{-727}{343}$	(2512, 9347, 13145, 1409)	18.306
77	$\frac{28}{53}$	$\left(\frac{-1379}{192144}, \frac{-62735}{192144}\right)$	$\frac{577723}{2579259}$	(111417913176, 652385155333, 3074200685543, 294747082303)	58.008

Table 1: 9 concrete primitive solutions spanning Height 4–14 digits. All n satisfy mod 5, 16, 29 conditions and gcd=1.

n	u	(x_0, y_0)	k	Solution (x, y, z, w) $(0 < x \leq y \leq z)$	height(k)
83	$\frac{-8}{29}$	$\left(\frac{19}{27}, \frac{20}{27}\right)$	$\frac{2087}{1741}$	(222601, 2408274, 2863999, 292549)	31.655
89	$\frac{-4}{9}$	$\left(\frac{1}{124}, \frac{-183}{124}\right)$	$\frac{-1944381}{9383}$	(1741159879, 278196472772, 415156380825, 38743789163)	51.797
89	$\frac{-4}{9}$	$\left(\frac{1}{124}, \frac{-183}{124}\right)$	$\frac{-1944381}{9383}$	(1741159879, 278196472772, 415156380825, 38743789163)	51.797
101	$\frac{64}{585}$	$\left(\frac{210919}{356203}, \frac{13332}{356203}\right)$	$\frac{4970540937918897947896533}{3317107705094098135090213 9}$	(115665246982780081781754947091286365446352306 99, 555334675493196846043214038366018613181974104 55, 962016944810071466651767339365023461011972497 72, 826451931756204573585191487789906540953263418 9)	218.043
119	$\frac{-133}{157}$	$\left(\frac{-506251}{59374565}, \frac{-130362878}{59374565}\right)$	$\frac{-5229646070084}{2135357527917}$	(30287965481668073, 434289126261707635, 996472927107738074, 77496631489680909)	82.860
123	$\frac{32}{37}$	$\left(\frac{-9}{22}, \frac{1}{22}\right)$	$\frac{-159}{89}$	(10778, 13981, 20717, 1671)	19.127
137	$\frac{56}{181}$	$\left(\frac{1309}{4936}, \frac{-1315}{4936}\right)$	$\frac{-15311}{18473}$	(28988, 107079, 369559, 26597)	28.398
141	$\frac{-60}{61}$	$\left(\frac{512045}{261178}, \frac{-964513}{261178}\right)$	$\frac{-19014301}{5702545}$	(72211175, 446853184, 1030134149, 73587999)	40.428
149	$\frac{-19}{85}$	$\left(\frac{-929}{1733}, \frac{3156}{1733}\right)$	$\frac{236}{383}$	(636, 835, 977, 77)	12.442
159	$\frac{4}{29}$	$\left(2, \frac{-3}{7}\right)$	$\frac{-1747}{2359}$	(337729, 353654, 622355, 43357)	25.795
161	$\frac{77}{81}$	$\left(\frac{-2063}{3319}, \frac{36}{3319}\right)$	$\frac{-292}{507}$	(262692, 641495, 842767, 60149)	24.866
173	$\frac{8}{9}$	$\left(\frac{-7}{5}, \frac{24}{35}\right)$	$\frac{-257}{42}$	(63816, 94031, 110677, 7997)	20.166
179	$\frac{-112}{145}$	$\left(\frac{17337}{532423}, \frac{-1046582}{532423}\right)$	$\frac{28727529}{1382477}$	(586876415, 2789777249, 4823616902, 311333271)	44.395
187	$\frac{84}{197}$	$\left(\frac{467}{12540}, \frac{5717}{12540}\right)$	$\frac{-4261}{5129}$	(6086, 9507, 10861, 761)	20.390
197	$\frac{-188}{137}$	$\left(\frac{-4538}{19963}, \frac{209465}{19963}\right)$	$\frac{-364547}{1019}$	(50704, 298617, 012059, 180461)	31.740
209	$\frac{4}{17}$	$\left(\frac{9}{22}, \frac{-1}{22}\right)$	$\frac{4210157}{2492579}$	(456741325525, 11021276613067, 13918037099988, 879530604163)	60.758
213	$\frac{-96}{85}$	$\left(\frac{2859}{2033}, \frac{6602}{2033}\right)$	$\frac{546386203}{285025889}$	(46410580396087, 48706449849644, 118968414820139, 6940828958469)	66.220
221	$\frac{-7}{13}$	$\left(\frac{5}{3}, \frac{92}{51}\right)$	$\frac{4554725986626873439777}{2251181595697275141261}$	(18170075829628946534152326312538321309945, 32469491979966057952627731170720804388014 1, 54594243497875426113235638376549635732788 8, 31804802854316895728509269912421630763669)	190.309
227	-1	$(1, -2)$	$\frac{613}{887}$	(31731, 855269, 2323538, 130273)	28.180
233	$\frac{4}{53}$	$\left(\frac{-4507}{888}, \frac{-5047}{888}\right)$	$\frac{413383}{450883}$	(1103010172, 1861529873, 3269841633, 185225597)	43.493

Table 2: 19 concrete primitive solutions spanning Height 3–42 digits. All n satisfy mod 5, 16, 29 conditions and gcd=1.

n	u	(x_0, y_0)	k	Solution (x, y, z, w) $(0 < x \le y \le z)$	height(k)
239	$\frac{136}{261}$	$\left(\frac{29531}{484199}, \frac{71760}{484199}\right)$	$\frac{-890704612273}{634303599357}$	(1640984772620303606, 3117491002540389405, 3477412467901354591, 215837293427890067)	85.474
249	$\frac{-24}{145}$	$\left(\frac{-73441}{147}, \frac{12272}{21}\right)$	$\frac{-49547964571}{43598116307}$	(7790928430539881312, 59531023402497214735, 61689237905359407721, 3842966957265123567)	93.180
251	$\frac{7}{9}$	$\left(\frac{-137}{187}, \frac{-138}{187}\right)$	$\frac{188}{43}$	(1403, 1530, 3109, 169)	14.875
383	$\frac{204}{377}$	$\left(\frac{-12212}{64707}, \frac{-38035}{64707}\right)$	$\frac{1550408297639667679}{1935402453467912411}$	(52271274597586749652046870748358, 56442182599777575561849640531969, 68037165015212162410398738968939, 3396464828358953323939105317063)	149.093
389	$\frac{989}{232}$	$\left(\frac{29825}{190557}, \frac{31732}{190557}\right)$	$\frac{59056815385513828}{47189577215495093}$	(2790220517471471853994808398379, 3375566679888983646972955355900, 3963453138704995482589945354391, 194958328042827044471167136157)	139.722
393	$\frac{56}{73}$	$\left(\frac{-209}{863}, \frac{-36}{863}\right)$	$\frac{-127457}{146606}$	(201277984, 2153997596, 3061641247, 142059643)	41.032
1003	$\frac{-16}{65}$	$\left(\frac{-48448}{308665}, \frac{-62703}{44095}\right)$	$\frac{1220947}{2671121}$	(357153561, 441486806, 522795199, 15911377)	39.595
1007	$\frac{-220}{157}$	$\left(\frac{321}{60268}, \frac{973057}{60268}\right)$	$\frac{-246903836004572641662901577}{6914579901675743188735327}$	(18754633151703759081272317852581751270809408588853, 24040798893695666175180959976981448725994130618754, 276962098995579830477463625832507108420988928415519, 7339332340035270076957220087297502594586408542833)	232.595
1013	$\frac{3}{101}$	$\left(\frac{20951}{17421}, \frac{5908}{17421}\right)$	$\frac{9963327853555}{2964128052389}$	(24574653502948757745404, 98778234488177851314697, 101516509859021233400459, 3148857129793207750683)	104.165

Table 3: 9 concrete primitive solutions spanning Height 3–51 digits. All n satisfy mod 5, 16, 29 conditions and gcd=1.

10 Conclusion

This yields rational points (x, y, t) of (3) via Mordell-Weil groups of (7). If we could find a rational number u and the associated elliptic curve $E_{n,u}$ with $rank E_{n,u} > 0$, then the method systematically generates infinte families of primitive solutions of (1) .

While a universal parametric solution for all n remains an open question, the successful derivation of a 46-digit solution for $n = 101$ through our elliptic curve reduction framework marks a significant departure from previous constraints. Future work will focus on analyzing the height of these points to better understand the arithmetic complexity of solutions for $n \equiv 2 \pmod 3$. While a universal parametric solution for all n remains an open question, the successful derivation of a 46-digit solution for $n = 101$ through our elliptic curve reduction framework marks a significant departure from previous constraints. Future work will focus on analyzing the height of these points to better understand the arithmetic complexity of solutions for $n \equiv 2 \pmod 3$.

Appendix: MAGMA Implementation

```
// search rational points for 4-descent fd and bound M

SetClassGroupBounds("GRH");

function RP4(fd,M)
 T0:=Realtime();
 for J:=1 to #fd do
  printf "J="; J;
    FD:=fd[J];
    pts:=PointsQI(FD,M);
    F,m:=AssociatedEllipticCurve(FD); F;
    printf "rootno="; RootNumber(F);
    for K:=1 to #pts do
      P:=m(pts[K]); P; printf "height "; Height(P);
      IsPoint(F,P[1]);
    end for; //K
 end for; //J
 T1:=Realtime(T0);
 printf "realtime="; T1;
 return #fd;
end function;

// calculate 4-descent for considering y^2=f(x) as 2-descent of the syzygy
// elliptic curve of 4th degree polynomial f(x)

P<x> := PolynomialRing(Rationals());
```

```
function C0(f)
SetClassGroupBounds("GRH");
C := HyperellipticCurve(f);
time fd := FourDescent(C : RemoveTorsion);
#fd;
return fd;
end function;

//
// Example for elliptic curve :
//   E_{41,u}: Y^2 = 287644205070868215349*X^4 - 1436879617899897971340*X^3  \
//                 + 3997420003135110578878*X^2 - 6003499258764516668940*X \
//                 + 4279348392455021754229
// Execute:
//
//   fd:=C0(287644205070868215349*x^4 - 1436879617899897971340*x^3  \
//          + 3997420003135110578878*x^2 - 6003499258764516668940*x \
//          + 4279348392455021754229);
//
//   RP4(fd,10^12);
//
```

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